PMM U.S.S.R.,Vol.47,No.1,pp.14-21,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

UDC 531.36

CONTINUOUS NORMAL FORM OF A CLASS OF NONAUTONOMOUS PARAMETRICALLY-PERTURBED SYSTEMS AND ITS APPLICATION*

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A problem on the continuous nomarlization of nonautonomous parametrically-perturbed systems of differential equations with a constant linear part is examined. A class of nonautonomous systems is delineated, which leads to a continuous normal form of resonance type under a formal transformation with continuous coefficients bounded with respect to a parameter and to time. The structure of the normal form of almost-periodic systems is detailed. The results obtained are applied to the study of the problem on the birth of stationary modes in a neighborhood of the resonance.

The continuous normal form was introduced earlier in /1,2/ for autonomous parametrically-perturbed systems and the problem of the change of stability as the system passes through resonance was studied. For nonautonomous systems of differential equations not dependent on parameters, the normal form has been studied for periodic systems /3–7/ as well as for systems whose coefficients are finite trigonometric sums /8,9/. General nonautonomous systems independent of parameters were analyzed in /10/. Methods developed in /5/ were generalized in /11/ to systems depending analytically on small parameters.

1. Statement of the problem. Preliminary results. Let $K^n(R^n)$ be an n-dimensional complex (real) vector space, P^n be the set of n-dimensional integral vectors $p=(p_1, \ldots, p_n)$, $|p|=|p_1|+\ldots+|p_n|$. If $p_s\geqslant 0$, then $p\in P_+^n$. Let $D\subset R^d$ be some closed d-dimensional domain. By C we denote the set of complex-valued functions $f(t,\mu)$ continuous and bounded in $R^1\times D$, $t\in R^2$, $\mu\in D$. In K^n we consider a differential equation system depending continuously on a parameter

$$z^\circ=A(\mu)z+\sum_{j=2}^{\infty}F^{(j)}(t,\mu,z) \tag{1.1}$$

where $F^{(j)}(t,\mu,z)$ is a jth-order vector-valued form in z, while its s-component is

$$F_s^{(j)}(t,\mu,z)=\sum_{|p|=j}f_p^{(s)}(t,\mu)z^p,\quad f_p^{(s)}(t,\mu)\in C \tag{1.2}$$

We assume that the $n\times n$-matrix $A(\mu)$ is reduced to Jordan form by a linear transformation continuous in D. The presentation is made only for the case when $A(\mu)=\operatorname{diag}(\rho_1(\mu),\ldots,\rho_n(\mu))$.

As an admissible class of transformations we shall examine the formal series

$$z=x+\sum_{j=2}^{\infty}\Phi^{(j)}(t,\mu,x) \tag{1.3}$$

in which the structure of the vector-valued form $\Phi^{(j)}$ is the same as in (1.1). To be precise, writing $\Phi_s^{(j)}$ as in (1.2), we take it that $\varphi_p^s(t,\mu)\in C$. We shall examine the simplest form of the system

$$x^\circ=A(\mu)x+\sum_{j=2}^{\infty}G^{(j)}(t,\mu,x) \tag{1.4}$$

to which system (1.1) is reducible by transformation (1.3). Let $g_p^{(s)}(t,\mu)$ be the coefficients of the s-th equation in (1.4). Taking $\varphi_p^{(s)}(t,\mu)$ and $g_p^{(s)}(t,\mu)$ as unknown, to determine them from the condition of reducibility of (1.1) to (1.4) we obtain the equation

$$\dot\varphi_p^{(s)}+\langle p-\delta_s,\rho(\mu)\rangle\varphi_p^{(s)}=v_p^{(s)}(t,\mu)-g_p^{(s)}(t,\mu) \tag{1.5}$$

where $v_p^{(s)}(t,\mu)$ is a known function of class C if all the preceding coefficients $\varphi_q^{(j)}$, $g_q^{(j)}$, $|q|<|p|$, are of class C, δ_s is the s-th unit vector, $\rho(\mu)=(\rho_1(\mu),\ldots,\rho_n(\mu))$, $\langle\,,\rangle$ is the scalar product. Thus, the problem posed is essentially related to the resolution of the question on the possibility of choosing functions $g_p^{(s)}(t,\mu)\in C$ such that Eq.(1.5) has a solution of the same class. If we succeed in choosing $g_p^{(s)}$ as constant in t, then system (1.4) is

*Prikl.Matem.Mekhan.,Vol.47,No.1,pp.27-36,1983 autonomous (and linear if all $g_p^{(s)} = 0$). It is comparatively easy to examine the case when $\operatorname{Re}\rho_s(\mu) > \alpha > 0$ or $\operatorname{Re}\rho_s(\mu) \leqslant \alpha < 0$) in D (for any s). In this case, for any $g_p^{(s)}(t, \mu) \in C$, (1.5) has a solution unique, bounded and continuous in $R^1 \times D$. When $g_p^{(s)} = 0$ this solution is

$$\varphi_p^{(s)}(t, \mu) = \int_0^t \exp[\langle p - \delta_s, \rho(\mu)\rangle(\tau - t)]\, v_p^{(s)}(\tau, \mu)\, d\tau \tag{1.6}$$

(The continuity in μ follows from the uniform convergence of the integral). Thus, in this noncritical case system (1.1) is reducible to a linear system.

We shall analyze the problem on the continuous normalization of system (1.1) in the more complex situation when in D there exists at least one point μ_0 at which at least one eigenvalue of matrix $A(\mu)$ has a zero real part. In this case we can find p such that $\operatorname{Re}\langle p - \delta_s, \rho(\mu)\rangle \to 0$ as $\mu \to \mu_0$, and for these p the solution (1.6), in general, ceases to exist at point μ_0. In connection with this the necessity arises of choosing $g_p^{(s)}(t, \mu) \in C$ such that in the situation described Eq. (1.5) has a solution from C. We assume the simplest choice $g_p^{(s)} = v_p^{(s)}$, but it does not yield the maximum simplification of system (1.1) in the class of transformations (1.3). Let us study in more detail an equation of form (1.5)

$$\varphi^{\cdot} = a(\mu)\varphi + w(t, \mu) \equiv a(\mu)\varphi + v(t, \mu) - g(t, \mu) \tag{1.7}$$

where the function $a(\mu) = \eta(\mu) + i\,k(\mu)$ is continuous in D. We separate D into subsets $D_0 = \{\mu \mid \eta(\mu) = 0\}$, $D_\pm = \{\mu \mid \eta(\mu) \gtrless 0\}$ and we introduce the functions

$$F_\pm(t, \mu) = \int_0^t e^{\mp a(\mu)\tau} w(\tau, \mu)\, d\tau, \qquad \mu \in D_\pm$$

$$F(t, \mu) = \int_0^t e^{-ik(\mu)\tau} w(\tau, \mu)\, d\tau, \qquad \mu \in D$$

For function $F_\pm(t, \mu)$ we introduce the mean value

$$M_\pm(\mu) = \lim_{T \to +\infty} \frac{1}{T}\int_0^T F_\pm(t, \mu)\, dt$$

Lemma 1.1. Let function $w(t, \mu) \in C$ be such that:
1) function $F(t, \mu)$ is bounded with respect to t in R^1 for all $\mu \in D$,
2) the mean

$$M(\mu) = \lim_{T \to +\infty} \frac{1}{2T}\int_{l-T}^{l+T} F(t, \mu)\, dt \tag{1.8}$$

exists uniformly in μ and t. Then, if $D_+ \cup D_- \neq \varnothing$, then a unique solution of Eq. (1.7) of class C exists, defined by the formula

$$\varphi(t, \mu) = e^{a(\mu)t}\left(\varepsilon(\mu) + \int_0^t e^{-a(\mu)\tau} w(\tau, \mu)\, d\tau\right) \tag{1.9}$$

where $\varepsilon(\mu) = \mp M_\pm(\mu)$, $\mu \in D_\pm$, $\varepsilon(\mu) = -M(\mu)$, $\mu \in D_0$.

Proof. We consider the general solution of Eq. (1.7), having the form (1.9) with an arbitrary constant $\varepsilon(\mu)$. From the lemma's condition 1) it follows that if $\mu \in D_0$, then (1.9) is a solution of class C for any $\varepsilon(\mu)$. If, however, $\mu \in D_\pm$, then for each fixed μ the Eq. (1.7) with any $w(t, \mu)$ has a solution bounded in R^1, which is determined by formula (1.9) with the following choice of $\varepsilon(\mu)$:

$$\varepsilon_+(\mu) = -\int_0^{+\infty} e^{-a(\mu)\tau} w(\tau, \mu)\, d\tau, \quad \mu \in D_+ \tag{1.10}$$

$$\varepsilon_-(\mu) = \int_{-\infty}^0 e^{-a(\mu)\tau} w(\tau, \mu)\, d\tau, \quad \mu \in D_-$$

In these equalities the functions $\varepsilon_\pm(\mu)$ are continuous in μ in the appropriate domains. By varying the value of $\varepsilon(\mu)$ in D_0, we can construct different solutions of Eq. (1.7), bounded in R^1. We convince ourselves that among them there exists a unique solution continuous in $\mu \in D$. The discontinuity of the solutions with respect to μ is connected with the behavior of $\varepsilon_\pm(\mu)$ as μ approaches parts of the boundary of sets $D_\pm$, belonging to D_0. We denote these parts by $\Gamma_\pm$. If $\mu_0 \in \Gamma_\pm$, then the integrals (1.10) do not exist in general.

The subsequent presentation relies on the method of Cesàro summation of improper integrals /12/, according to which the improper integral

$$\int_0^{+\infty} f(t)\,dt$$

is ascribed the value

$$M\{\Phi\} = \lim_{T\to+\infty} \frac{1}{T}\int_0^t \Phi(t)\,dt, \quad \Phi(t) = \int_0^t f(\tau)\,d\tau$$

If the original integral converges, then

$$\int_0^{+\infty} f(t)\,dt = M\{\Phi\}$$

(regularity of the Cesàro method). Using the method mentioned, we convince ourselves first of all that the function $F_\pm(t,\mu)$ in $D_\pm$ has a mean

$$M_\pm(\mu) = \mp\varepsilon_\pm(\mu) \tag{1.11}$$

continuous in μ, which follows from the convergence of integrals (1.10). We ascertain the behavior of function $M_+(\mu)$ as μ approaches Γ_+. Summing the function $e^{-ik(\mu)\tau}w(\tau,\mu)$ for $\tau \in [0;+\infty)$, we obtain (uniformly in μ by virtue of condition 2)

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^t F(\tau,\mu)\,d\tau = M(\mu) \tag{1.12}$$

Now fixing the point $\mu_0 \in \Gamma_+$, we have

$$\lim_{\mu\to\mu_0} M_+(\mu) = \lim_{\mu\to\mu_0}\lim_{T\to+\infty} \frac{1}{T}\int_0^T F_+(\tau,\mu)\,d\tau = \lim_{T\to+\infty}\lim_{\mu\to\mu_0} \frac{1}{T}\int_0^T F_+(\tau,\mu)\,d\tau = M(\mu_0) \tag{1.13}$$

When computing the limit we used the continuity of $F_+(\tau,\mu)$ and the uniformity in μ of limit (1.12), permitting us to interchange the order of the limit passages. Further, let us consider the function $\varepsilon(\mu)$ on the boundary Γ_-. We write the second relation in (1.10) as

$$M_-(\mu) = \int_0^{+\infty} e^{a(\mu)\tau} w(-\tau,\mu)\,d\tau$$

For $\mu^0 \in \Gamma_-$ we have

$$\lim_{\mu\to\mu_0} M_-(\mu) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \left(\int_0^t e^{ik(\mu_0)\tau} w(-\tau,\mu_0)\,d\tau\right) dt = -M(\mu_0) \tag{1.14}$$

When computing the limit, besides the interchange of limit passages, we have used the equality of means $M\{f(-t)\} = M\{f(t)\}$ resultant from (1.8). It follows from (1.11)–(1.14) that by setting $\varepsilon(\mu) = \mp M_\pm(\mu)$ when $\mu \in D_\pm$ and $\varepsilon(\mu) = M(\mu)$ when $\mu \in D_0$, we obtain the unique solution (1.9) of class C.

Note. The proof given relies essentially on the Cesàro summation method. Precisely because of this the lemma's main requirement is connected with the assumption of existence of a mean value. This requirement is satisfied, in particular, by periodic and almost-periodic systems to which we shall henceforth apply Lemma 1.1. By using other regular methods of computing the values of divergent integrals, we can obtain other conditions for the existence of solutions of class C of (1.7). However, we do not dwell on this here.

Lemma 1.1 enables us to delineate a class of nonautonomous systems for which a continuous normal form can be constructed. From the presentation we see that the system (1.4) corresponding to it will contain only resonance terms x^p in which p satisfies the condition

$$(\exists\mu_0 \in D)\ (\mathrm{Re}\,\langle p - \delta_s, \rho(\mu_0)\rangle = 0)$$

2. Continuous normal form of almost-periodic systems. We consider system (1.1) under the assumption that all the coefficients $f_p^{(s)}(t,\mu)$ are almost-periodic functions of t uniformly with respect to μ in D, and we denote the set of all such functions by C_0 /13,14/. All bounded solutions of Eq.(1.7) with almost-periodic inhomogeneities are almost-periodic functions; therefore, the problem, analyzed in Sect.1, of the transformation of system (1.1) to simplest form in the class of transformations (1.3) with coefficients from C is equivalent for almost-periodic systems to the same problem but in the class of the transformation of (1.1) with almost-periodic coefficients. Lemma 1.1 yields the conditions for the existence of a solution of Eq.(1.7), continuous in D. Let us dwell on the question of choosing in (1.7) a function $g(t, \mu) \in C_0$, guaranteeing the fulfillment of the conditions of Lemma 1.1. We consider Eq. (1.7) wherein $v(t, \mu) \in C_0$. Let $S_v = \{\lambda_n\}$ be the spectrum of function v and $S_v^k(\mu) = \{\lambda_n - k(\mu)\}$ be the displaced spectrum.

Definition 2.1. The Fourier exponent λ_n is called a resonance exponent if $(\exists \mu_0 \in D_0)$ $(\lambda_n - k(\mu_0) = 0)$.
In accord with the definition we separate S_v into resonance (R_v) and nonresonance (H_v) parts. To them correspond analogous subsets $R_v^k(\mu)$, $H_v^k(\mu)$ in the displaced spectrum. We say that Eq. (1.7) is of type F if the nonresonance part of the displaced spectrum of function $v(t, \mu)$ is nonzero uniformly in μ

$$(F)\ (\exists \alpha > 0)\ (\forall \mu \in D_0)\ (\forall \lambda_n \in H_v)\ (|\lambda_N - k(\mu)| \geqslant \alpha)$$

We introduce into consideration a function $v_\alpha(t, \mu)$ which we define as follows. Let α be a sufficiently small fixed number. On the complex plane K^2 we consider α, i.e., a neighborhood of zero $U_\alpha(0)$, the displaced spectrum $S_v^k(\mu)$ and the set

$$S_{v,\alpha}^k(\mu) = \bigcup_{\mu \in D_0} (S_v^k(\mu) \cap U_\alpha(0))$$

To this set corresponds an analogous part $S_{v,\alpha}$ of spectrum S_v. The whole resonance part of spectrum S_v is contained in $S_{v,\alpha}$, but the latter can contain as well a part of the nonresonance set H_v if H_v has a limit point in R_v. By $v_\alpha(t, \mu)$ we denote the "α-cut" of function $v(t, \mu)$, i.e., a nearly periodic function whose spectrum coincides with $S_{v,\alpha}$, while the Fourier coefficients coincide with the corresponding Fourier coefficients of function $v(t, \mu)$. The function $v(t, \mu) - v_\alpha(t, \mu)$ satisfies condition (F) and belongs to class C_0.

Lemma 2.1. Let D be a closed bounded set in $R^d, D_+ \cup D_- \neq \varnothing$, and let the function $v(t, \mu) \in C_0$ satisfy, together with $a(\mu)$, a Lipschitz condition in μ. Then Eq. (1.7) with

$$g(t, \mu) = v_\alpha(t, \mu) \tag{2.1}$$

has a unique solution $\varphi(t, \mu) \in C_0$. If (1.7) is an equation of type F, then

$$g(t, \mu) = \sum_{\lambda \in R_v} v_\lambda(\mu) e^{i\lambda t} \tag{2.2}$$

where $v_\lambda(\mu)$ are the Fourier coefficients of function $v(t, \mu)$ and R_v is the resonance part of this function's spectrum. The solution $\varphi(t, \mu)$ is determined by formula (1.9), has the spectrum H_v, satisfies a Lipschitz condition and has a mean value equal to zero.

The proof (which we do not carry out in detail) consists in verifying the fulfillment of the conditions of Lemma 1.1. When $g = v_\alpha$ (when condition (F) is fulfilled this equality reduces to (2.2)), with the aid of the Favard theorem /15/ we convince ourselves of the fulfillment of condition 1 of Lemma 1.1 and of the fact that $F(t, \mu) \in C_0$. The latter ensures the existence for any fixed mean value (1.8). The uniformity with respect to μ of the limit in (1.8) is ensured by the Lipschitz condition. Next, we can establish that for the values of $\varepsilon(\mu)$ indicated in formula (1.9) the solution's spectrum coincides with H_v, while the mean value equals zero.

We pass on to the question of the normalization of a uniformly almost-periodic system (1.1) whose coefficients $f_p^{(s)}(t, \mu)$ satisfy a Lipschitz condition in μ. Let $S_{p,s}$ be the spectrum of coefficient $f_p^{(s)}(t, \mu)$. We introduce the sets: S_j, the system's jth-order spectrum; S_2^k, the system's spectrum in the k-th approximation; and S_2^∞, the system's spectrum, i.e.,

$$S_j = \bigcup_{s=1}^{n} \bigcup_{|p|=j} S_{p,s}, \qquad S_2^k = \bigcup_{j=2}^{k} s_j$$

By $N_2^k(N_2^\infty)$ we denote the minimum modulus of set $S_2^k(S_2^\infty)$. The elements $\gamma \in N_2^k$ have the representation (r_j are integers)

$$\gamma = \sum r_j \lambda_j, \qquad \lambda_j \in S_2^k \tag{2.3}$$

In N_2^k we pick out a subset of those elements γ for which in (2.3) $\Sigma |r_j| \leqslant k$, $\gamma \in N_2'^k$.

Definition 2.2. System (1.1) has a kth-order internal resonance at a point $\mu_0 \in D$ if there exists such an integral vector $q \in P^n$ with relatively prime components q_j and $|q| = |q_1| + \ldots + |q_n| = k$, such that

$$i\langle q, \rho(\mu_0)\rangle \in N_2'^k$$

(For ω-periodic systems $N_2^\infty = \{2k\pi\omega^{-1}\}$, while for autonomous systems $N_2^\infty = \{0\}$, and for a fixed μ we arrive at the usual definition of internal resonance in such systems).

Definition 2.3. The vector $p \in P_+^n$, $|p| = k$, and the corresponding terms of the s-th equation in system (1.1), are said to be resonance in D if

$$(\exists\mu_0 \in D)\,(i\langle p-\delta_s, \quad \rho(\mu_0)\rangle \in N_2'^{k})$$

where δ_s is the s-th unit vector in R^n. The set of all resonance vectors of the s-th equation is denoted $L_D^{(s)}$.

Definition 2.4. System (1.1) is called an F-system if for any resonance vector p, $|p|=k$, the point $i\langle p-\delta_s, \rho(\mu_0)\rangle$ is not a limit point of set $N_2{}^k$.

For the normalization of system (1.1) with the use of transformation (1.3) we should successively solve Eq. (1.5)

$$\varphi_p^{\cdot(s)} = -\langle p-\delta_s, \rho(\mu)\rangle \varphi_p^{(s)} + v_p^{(s)}(t,\mu) - g_p^{(s)}(t,\mu)$$

with the aid of Lemma 2.1, confining ourselves to the following alternatives:

1) if p is a nonresonance vector, then in (1.5) we set $g_p^{(s)}(t,\mu)=0$ and from (1.5) we find $\varphi_p^{(s)}(t,\mu)$ as the unique almost-periodic solution of class C_0;

2) if p is a resonance vector, then we set $g_p^{(s)}(t,\mu) = v_{p,\alpha}^{(s)}(t,\mu)$; here again $\varphi_p^{(s)}(t,\mu)$ is found as the unique solution of class C_0.

We note that since when solving Eq. (1.5) the spectrum of function $\varphi_q^{(j)}$ coincides with the nonresonance part of the spectrum of $v_q^{(j)}(t,\mu)$, when solving Eqs. (1.5) successively the spectrum of function $v_p^{(s)}(t,\mu)$, depending on $\varphi_q^{(j)}(t,\mu)$ $|q|<|p|$, is contained in $N_2'^{k}$, where $k=|p|$. Precisely this is taken into account in Definition 2.3. If (1.1) in an F-system, then each equation in (1.5) satisfies condition (F) and the selection of $g_p^{(s)}(t,\mu)$ simplifies

$$g_p^{(s)}(t,\mu) = \sum_{\lambda\in R_{v_p^{(s)}}} v_{p,\lambda}^{(s)}(\mu)\, e^{i\lambda}$$

Summarizing the above presentation, we arrive as the following statement.

Theorem 2.1. Let system (1.1) be uniformly almost-periodic in a closed bounded domain D and let $\rho_s(\mu)$, $f_p^{(s)}(t,\mu)$ satisfy a Lipschitz condition in μ. Then a transformation (1.3) with uniformly almost-periodic coefficients, continuous in $\mu\in D$, exists leading system (1.1) to the continuous normal form

$$x_s^{\cdot} = \rho_s(\mu)\,x_s + \sum_{p\in L_D^{(s)}} v_{p,\alpha}^{(s)}(t,\mu)\,x^p \tag{2.4}$$

If (1.1) is an F-system, then (2.4) becomes

$$x_s^{\cdot} = \rho_s(\mu)\,x_s + \sum_{p\in L_D^{(s)}} \Big(\sum_{\lambda\in R_{v_p^{(s)}}} v_{p,\lambda}^{(s)}(\mu)\,e^{i\lambda t}\Big)\,x^p$$

where $R_{v_p^{(s)}}$ is the resonance part of the sepctrum of functions $v_p^{(s)}(t,\mu)$.

From the theorem we see that the continuous normal form of resonance systems are autonomous only if (1.1) is an F-system and $R_{v_p^{(s)}}=\{0\}$. The latter is fulfilled obviously if in (1.1) there is only the identity resonance and, possibly, an internal resonance of the form

$$\langle m, \rho(\mu_0)\rangle = 0, \quad \mu_0\in D_0, \quad m\in P^n$$

Let the F-system have l pairs of purely imaginary eigenvalues at the point μ_0, while the remaining eigenvalues $\rho_j(\mu)$ have negative real parts. By $\sigma_s(\mu)\pm i\nu_s(\mu)$ we denote the critical eigenvalues, $\sigma_s(\mu_0)=0$, $\nu_s(\mu_0)\neq 0$, $s=1,\ldots,l$. Let D be a neighborhood of point μ_0. We take it that the kth-order internal resonance

$$\langle m, \nu(\mu_0)\rangle = \lambda\in N_2'^{k}, \quad |m|=k \tag{2.5}$$

exists at point μ_0. Taking (2.5) as the unique resonance in D, we write out the continuous normal form in the situation described. We represent the vector x as a triple of vectors $x=(u,\bar u, w)$, where u is an l-dimensional complex vector corresponding to the critical variables and w is an $n-2l$-dimensional vector. The original real system (1.1) is written in the variables $u,\bar u, w$ in the following manner:

$$u_s^{\cdot} = (\sigma_s(\mu)+i\nu_s(\mu))\,u_s + u_s\sum_{|p|=1}^{\infty} v_p^{(s)}(\mu)\,\omega^p + \sum_{(p,q)\in R_s} v_{p,q}^{(s)}(\mu)\exp(-i\chi_{p,q}^{(s)}\lambda t)\,u^p\bar u^q \tag{2.6}$$

$$w_j^{\cdot} = \rho_j(\mu)\,w_j + w_j\sum_{|p|=1}^{\infty} g_p^{(j)}(\mu)\,\omega^p + w_j\sum_{(p,q)\in Q} g_{p,q}^{(j)}(\mu)\exp(-i\varkappa_{p,q}\lambda t)\,u^p\bar u^q$$

$$p,q\in P_+^n, \quad \omega=(\omega_1,\ldots,\omega_l), \quad \omega_s=u_s\bar u_s$$

Here R_s is the set of resonance pairs (p, q) satisfying the equation $p - q - \delta_s = \chi^{(s)}_{p,q}\, m$, where $\chi^{(s)}_{p,q}$ is any integer, Q is the set of resonance pairs satisfying the condition $p - q = \varkappa_{p,q} m$ with some integer $\varkappa_{p,q}$.

If $\lambda = 0$, then (2.6) is an autonomous system coinciding in structure with the normal form of autonomous systems. For such systems, not depending on a parameter, the expanded notation exists in /16/. When $\lambda \neq 0$ the system is nonautonomous. However it is not difficult to obtain an autonomous system if we introduce the substitution

$$u_s = r_s \exp\left[i\left(\nu_s(\mu_0)\, t + \varphi_s\right)\right]$$

with new real variables r_s, φ_s. After manipulations we have

$$\begin{aligned}
&r_s^{\cdot} = \sigma_s(\mu) r_s + r_s \sum_{|p|=1}^{\infty} \alpha_p^{(s)}(\mu) r^{2p} + \sum_{R_s} \alpha^{(s)}_{p,q}(\mu) \sin\left(\chi^{(s)}_{p,q}\psi - \theta^{(s)}_{p,q}\right) r^{p+q} \qquad (2.7)\\
&\varphi_s^{\cdot} = \Delta_s(\mu) + \sum_{|p|=1}^{\infty} \beta_p^{(s)}(\mu) r^{2p} + \sum_{R_s} \alpha^{(s)}_{p,q} \cos\left(\chi^{(s)}_{p,q}\psi - \theta^{(s)}_{p,q}\right) r^{p+q}\\
&w_j^{\cdot} = \rho_j(\mu) w_j + w_j \sum_{|p|=1}^{\infty} g_p^{(j)}(\mu) r^{2p} + w_j \sum_{Q} g^{(j)}_{p,q}(\mu)\, r^{p+q}\\
&\alpha_p^{(s)} + i\beta_p^{(s)} = v_p^{(s)}, \quad \alpha^{(s)}_{p,q} = |v^{(s)}_{p,q}|, \quad \Delta_s(\mu) = \nu_s(\mu) - \nu_s(\mu_0)\\
&\psi = \langle m, \varphi\rangle, \quad \varphi = (\varphi_1, \ldots, \varphi_l), \quad \sin\theta^{(s)}_{p,q} =\\
&\quad -|v^{(s)}_{p,q}|^{-1} \operatorname{Re} v^{(s)}_{p,q}, \quad \cos\theta^{(s)}_{p,q} = |v^{(s)}_{p,q}|^{-1} \operatorname{Im} v^{(s)}_{p,q}
\end{aligned}$$

Taking into account that in the first group of equations the variables φ_s occur only in the combination $\psi = \langle m, \varphi\rangle$, then, by introducing in the place of one of the equations for φ_s the equation for ψ, from system (2.7) we can separate out an $l+1$-dimensional subsystem in which the equations for $r_s (s = 1, \ldots, l)$ and ψ occur. System (2.7) can be used to study a number of problems of the bifurcation type. For example, it is possible to study the change in stability of neutral systems when passing through resonance. In an autonomous system a detailed analysis was made in /1,2/. Below we dwell on another bifurcation problem.

3. Birth of stationary modes in a neighborhood of third-order resonance.

We shall reckon that the four-dimensional system (1.1) has in domain D two pairs of eigenvalues $\sigma_s(\mu) \pm i\nu_s(\mu)$ such that $\sigma_s(\mu_0) = 0$, $\mu_0 \in D$, and that the resonance

$$\nu_1(\mu_0) - 2\nu_2(\mu_0) = \lambda \in N_2'^3 \qquad (3.1)$$

is realized. We take it that D is a neighborhood of μ_0. By D^* we denote the deleted neighborhood of point μ_0 and we let $\sigma_s(\mu_0) \neq 0$, $\forall\mu \in D$. In complex-conjugate variables system (1.1) is

$$z_s^{\cdot} = (\sigma_s(\mu) + i\nu_s(\mu))\, z_s + Z_s^{(2)}(\mu, z, \bar{z}, t) + \ldots \qquad (3.2)$$

The resonance terms in (3.2) are the following: $f_{0200}{}^{(1)} z_2{}^2$ in the first equation, $f_{1001}{}^{(2)} z_1 \bar{z}_2$ in the second. Continuous normalization up to third order takes (3.2) (under the condition that (1.1) is an F-system) to the form

$$\begin{aligned}
&u_1^{\cdot} = (\sigma_1(\mu) + i\nu_1(\mu))\, u_1 + a_1(\mu)\, e^{i\lambda t} u_2{}^2 + O(\|u\|^3) \qquad (3.3)\\
&u_2^{\cdot} = (\sigma_2(\mu) + i\nu_2(\mu))\, u_2 + a_2(\mu)\, e^{-i\lambda t} u_1 \bar{u}_2 + O(\|u\|^3)\\
&a_1(\mu) = M\{f^{(1)}_{0200} e^{-i\lambda t}\}, \quad a_2(\mu) = M\{f^{(2)}_{1001} e^{i\lambda t}\}
\end{aligned}$$

We introduce the variables r_s, φ_s and the small parameter ε by setting

$$u_s = \varepsilon r_s \exp\left[i\left(\nu_s(\mu_0)\, t + \varphi_s\right)\right]$$

By picking out in the new system of type (2.7) the three-dimensional subsystem $r_1, r_2, \psi = \varphi_1 - 2\varphi_2$, after the scale transformation $\rho_1 = \alpha_2 r_1$, $\rho_2 = \sqrt{\alpha_1\alpha_2}\, r_2$, where $\alpha_s(\mu) = |a_s(\mu)|$, we obtain

$$\begin{aligned}
&\rho_1^{\cdot} = \sigma_1(\mu)\, \rho_1 + \varepsilon s_1(\psi)\, p_2{}^2 + O(\varepsilon^2) \qquad (3.4)\\
&\rho_2^{\cdot} = \sigma_2(\mu)\, \rho_2 + \varepsilon s_2(\psi)\, \rho_1\rho_2 + O(\varepsilon^2)\\
&\psi^{\cdot} = \delta(\mu) + \varepsilon\left(c_1(\psi)\, \rho_1{}^{-1} \rho_2{}^2 + 2c_2(\psi)\, \rho_1\right) + O(\varepsilon^2)\\
&(\delta(\mu) = \nu_1(\mu) - 2\nu_2(\mu) - \lambda, \; s_i(\psi) = \sin(\psi - \theta_i),\\
&c_i(\psi) = \cos(\psi - \theta_i), \; \sin\theta_i = -\alpha_i{}^{-1} \operatorname{Re} a_i,\\
&\cos\theta_i = (-1)^{i-1} \alpha_i{}^{-1} \operatorname{Im} a_i, \; i = 1, 2)
\end{aligned}$$

At the resonance point $\mu = \mu_0$, where $\sigma(\mu_0) = \delta(\mu_0) = 0$, the stability analysis is completely analogous to that of autonomous systems /16/. System (3.4) is unstable when $\mu = \mu_0$ if $\theta_1(\mu_0) \neq \theta_2(\mu_0) + \pi$. Moreover, in the first approximation with respect to ε it has a partial solution in the form of "an unstable ray"

$$\psi = \psi_0, \; \rho_1 = s_{20}{}^{-1} z(t), \; \rho_2 = (s_{10}s_{20})^{-1/2} z(t), \; s_{i0} = s_i(\psi_0)$$

where ψ_0 is a root of the equation

$$\operatorname{ctg}(\psi - \theta_1(\mu_0)) + 2\operatorname{ctg}(\psi - \theta_2(\mu_0)) = 0 \tag{3.5}$$

from the interval $(\theta_2(\mu_0), \theta_1(\mu_0) + \pi)$ (we reckon that $\theta_1(\mu_0) < \theta_2(\mu_0)$), while $z(t)$ satisfies the equation $z^\cdot = \varepsilon z^2$. Let us consider system (3.4) in domain D^*. We shall seek the stationary modes of system (3.4), generated as μ passes into the domain D^*. Rejecting in (3.4) nonlinearities $O(\varepsilon^2)$, we consider the amplitude equations (the right-hand sides of (3.4) equated to zero). We take it that $\sigma_s(\mu), \delta(\mu)$ are of the order of smallness of ε. This assumptions follows in natural fashion from the continuity of $\sigma_s(\mu), \delta(\mu)$ and the smallness of dimain D. Seeking the positive solutions of the system of amplitude equations, first of all we find ρ_1 and ρ_2 from the first two equations; next, from the third equation we obtain

$$\sigma_1(\mu)\operatorname{ctg}(\psi - \theta_1(\mu)) + 2\sigma_2(\mu)\operatorname{ctg}(\psi - \theta_2(\mu)) = \delta(\mu) \tag{3.6}$$

Let σ_1 and σ_2 be of one sign. Then Eq.(3.6) in the interval $(\theta_2, \theta_1 + \pi)$ (as before we reckon that $\theta_1(\mu) < \theta_2(\mu)$) has a root $\psi = \psi_1^*(\mu)$ such that $s_i(\psi_1^*) > 0$ and a root $\psi = \psi_2^* = \psi_1^* + \pi$, but $s_i(\psi_2^*) < 0$.

We represent domain D^* as $D^* = D_{++}^* \cup D_{+-}^* \cup D_{-+}^* \cup D_{--}^*$, where the first sign corresponds to the sign of σ_1 and the second, of σ_2. From what we have presented it follows that the steady-state mode

$$\psi = \psi_1^*, \ \rho_1 = \rho_1(\psi_1^*), \ \rho_2 = \rho_2(\psi_1^*) \tag{3.7}$$

is generated in the domain D_{--}^*. The mode

$$\psi = \psi_2^*, \ \rho_1 = \rho_1(\psi_2^*), \ \rho_2 = \rho_2(\psi_2^*) \tag{3.8}$$

exists in domain D_{++}^*. We see that for a fixed ε and as $\mu \to \mu_0$ we have $\rho_i(\psi_1^*), \rho_i(\psi_2^*) \to 0$. When $\sigma_1 \sim \sigma_2$, $\delta = o(\sigma_s)$ the roots ψ_1^*, ψ_2^* tend to the corresponding root of Eq.(3.5): $\psi_1^* \to \psi_0$, $\psi_2^* \to \psi_0 + \pi$. In other words, in this situation the generation of the steady-state mode (3.7) takes place close to the unstable ray existing in the system when $\mu = \mu_0$. Analysis of Eq.(3.6) in the domains D_{+-}^* and D_{-+}^* shows that it does not always have a solution. The existence of a solution depends upon the relations between $\sigma_1, \sigma_2, \delta$. This case is not examined here in more detail.

Let $\rho_1^*, \rho_2^*, \psi^*$ be some stationary resonance mode (in particular, (3.7) or (3.8)), i.e., ψ^* is some root of Eq.(3.6), for which the ρ_1^*, ρ_2^* obtained from the amplitude equations are positive. Let us consider the question of the stability of this mode. Having set up the system of variational equations, we consider its characteristic equation

$$\begin{gathered}\gamma^3 - 2(\sigma_1 + \sigma_2)\gamma^2 + [\sigma_1^2 + \sigma_1^2 \operatorname{ctg}^2(\psi^* - \theta_1) - \\ 4\sigma_1\sigma_2 \operatorname{ctg}(\psi^* - \theta_1)\operatorname{ctg}(\psi^* - \theta_2)]\gamma - H = 0 \\ H = -2\sigma_1\sigma_2(\sigma_1 + 2\sigma_2 + \sigma_1\operatorname{ctg}^2(\psi^* - \theta_1) + 2\sigma_2\operatorname{ctg}^2(\psi^* - \theta_2))\end{gathered}$$

An application of the necessary stability conditions (Stodola's theorems) shows that modes (3.7) and (3.8), for which $\sigma_1\sigma_2 > 0$, are unstable. To detect in the system the resonance modes of the form $\psi = \psi^*, \rho_1 = \rho_1^*, \rho_2 = 0$ we should consider as well the values of μ for which $\sigma_s(\mu), \delta(\mu) = O(\varepsilon^2)$, and carry out one further step of the normalization. The latter is connected with the fact that when considering the amplitude equations to within ε, $\rho_1 = 0$ follows from $\rho_2 = 0$.

The author thanks V.V. Rumiantsev for attention to the work.

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Translated by N.H.C.